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On Two Pairs of Simultaneous Dual Integral Equations

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SUMMARY

This paper considers a system of coupled pairs of dual integral equations with constant coefficients involving Bessel functions of orders zero and unity. A solution is obtained in terms of the coefficients by reducing the system to a single integral equation of the Wiener–Hopf type with both the sum and difference kernels present.

A simple transformation of the system causes the coefficient of the sum kernel to vanish. The transformation leaves the Wiener–Hopf equation unaltered except for the coefficients which become complex. An equation of this type was solved by Spence in 1967. Although Spence's solution does not cover complex coefficients it can be modified to do so. The result is quoted in this paper and is used to solve the system of coupled pairs of dual integral equations of the present paper.

The adhesive contact problem recently solved by Gladwell is one in which the solution technique of the present paper has proved useful.

1. Introduction

Mixed boundary value problems are often solved by integral transform techniques. The approach leads to one or more pairs of dual integral equations. Some work has been done on simultaneous dual integral equations by Erdogan and Bahar [1], Westmann [2], Keer [3], Spence [4] and Khadem [5]. Erdogan and Bahar reduced the problem to a system of infinite linear algebraic equations. Westmann gave a closed form solution for the system involving Bessel functions whose orders differ by two: Keer has recently obtained a solution for coupled pairs of dual integral equations by using the operator notation of Erdelyi and Sneddon in conjunction with Westmann's method. Neither Westmann's solution nor Keer's, however, covers the system treated here. Spence and Khadem solved the system of the present paper for special values of the eight constant coefficients. It happens that other values of the coefficients introduce additional terms in the analysis which require the present treatment.

In the present paper, solution is given for the system of two pairs of dual integral equations

$$\begin{aligned} &\int_0^\infty t^{-1}[a_1\varphi_1(t)+a_2\varphi_2(t)]\,J_0(\rho t)\,dt=f_1(\rho), && \rho<1 \\ &\int_0^\infty [a_3\varphi_1(t)+a_4\varphi_2(t)]\,J_0(\rho t)=f_2(\rho), && \rho<1 \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\int_0^\infty t^{-1}[b_1\varphi_1(t)+b_2\varphi_2(t)]\,J_1(\rho t)\,dt=g_1(\rho), && \rho<1 \\ &\int_0^\infty [b_3\varphi_1(t)+b_4\varphi_2(t)]\,J_1(\rho t)\,dt=g_2(\rho), && \rho>1 \end{aligned} \tag{1.2}$$

subject to certain restrictions mentioned in the sequel, where a_i, b_i are known coefficients, real or complex. Equations (1.1) and (1.2) are denoted by system I. They are reduced in [5] to the following integral equation of the Wiener–Hopf type with both the sum and the difference kernels present:

$$s(t)=\frac{1}{\omega}r(t)+\frac{\beta_1}{\omega}\int_0^\infty s(\xi)k(t+\xi)d\xi+\frac{\beta_2}{\omega}\int_0^\infty s(\xi)k(t-\xi)d\xi\,,\ t>0 \tag{1.3}$$

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where

$$s(t) = t^{-1}\varphi_2(t)\,, \qquad k(t) = \sin(t)/\pi t$$
$$r(t) = \frac{2}{\pi}\left\{\frac{1}{a_1}H_1(t) + \frac{1}{a_3}H_2(t) - \frac{1}{b_1}D_1(t) - \frac{1}{b_3}D_2(t)\right\}, \tag{1.4}$$

wherein

$$H_1(t) = \int_0^1 f_1^*(\rho)\cos\rho t\,d\rho\,, \qquad H_2(t) = \int_1^\infty f_2^*(\rho)\cos\rho t\,d\rho$$
$$D_1(t) = \int_0^1 g_1^*(\rho)\sin\rho t\,d\rho\,, \qquad D_2(t) = \int_1^\infty g_2^*(\rho)\sin\rho t\,d\rho\,. \tag{1.5}$$

The starred functions are defined in terms of $f_i(\rho)$ and $g_i(\rho)$, thus

$$f_1^*(\rho) = \frac{d}{dx}\int_0^x \rho(x^2-\rho^2)f_1(\rho)d\rho\,, \qquad f_2^*(\rho) = \int_x^\infty \rho(\rho^2-x^2)^{-\frac{1}{2}}f_2(\rho)d\rho$$
$$g_1^*(\rho) = \frac{1}{x}\frac{d}{dx}\int_0^x \rho^2(x^2-\rho^2)^{-\frac{1}{2}}g_1(\rho)d\rho\,, \qquad g_2^*(\rho) = x\int_x^\infty (\rho^2-x^2)^{-\frac{1}{2}}g_2(\rho)d\rho\,. \tag{1.6}$$

The coefficients β_1, β_2 and ω occurring in (1.3) are

$$\beta_1 = \frac{b_4}{b_3} - \frac{b_2}{b_1} + \frac{a_4}{a_3} - \frac{a_2}{a_1}\,, \qquad \beta_2 = \frac{b_2}{b_1} - \frac{b_4}{b_3} + \frac{a_4}{a_3} - \frac{a_2}{a_1}$$
$$\omega = \frac{a_4}{a_3} - \frac{b_4}{b_3}\,, \qquad a_4 b_3 \neq a_3 b_4 \tag{1.7}$$

In section 2 it is argued that if system I is solved for φ_1 and $\varphi_2+\delta\varphi_1$, instead of φ_1 and φ_2, then the coefficient of the sum kernel can be made to vanish by an appropriate choice of the parameter δ. The parameter δ effectively transforms system I to the equivalent system II involving the unknown functions $\psi_1(t)$ and $\psi_2(t)$, where

$$\psi_1(t) = \varphi_1(t)\,, \quad \psi_2(t) = \varphi_2(t)+\delta\varphi_1(t)\,. \tag{1.8}$$

System II obviously reduces to the Wiener–Hopf equation with the sum kernel absent. The solution of this equation is used in section 3 to solve the original system for φ_1 and φ_2. The solution is shown in section 4, to agree in special cases with those of Spence [4], Khadem [5] and Gladwell [6].

2. The Equivalent System

Let $\psi_1(t)$ and $\psi_2(t)$ be defined as in (1.8), then the equivalent system becomes

$$\int_0^\infty t^{-1}[A_1\psi_1(t)+A_2\psi_2(t)]\,J_0(\rho t)dt = f_1(\rho) \qquad \rho<1$$
$$\int_0^\infty [A_3\psi_1(t)+A_4\psi_2(t)]\,J_0(\rho t)dt = f_2(\rho) \qquad \rho>1 \tag{2.1}$$

$$\int_0^\infty t^{-1}[B_1\psi_1(t)+B_2\psi_2(t)]\,J_1(\rho t)dt = g_1(\rho) \qquad \rho<1$$
$$\int_0^\infty [B_3\psi_1(t)+B_4\psi_2(t)]\,J_1(\rho t)dt = g_2(\rho) \qquad \rho>1 \tag{2.2}$$

where

$$A_1 = a_1-\delta a_2\,; \quad A_2 = a_2\,; \quad A_3 = a_3-\delta a_4\,; \quad A_4 = a_4$$
$$B_1 = b_1-\delta b_2\,; \quad B_2 = b_2\,; \quad B_3 = b_3-\delta b_4\,; \quad B_4 = b_4\,. \tag{2.3}$$

The system reduces to the following equation, which is equivalent to (1.3):

$$\Omega(t) = \frac{1}{\omega'}C(t) + \frac{\beta'_1}{\omega'}\int_0^\infty \Omega(\xi)k(t+\xi)d\xi + \frac{\beta'_2}{\omega'}\int_0^\infty \Omega(\xi)k(t-\xi)d\xi \tag{2.4}$$

for $t>0$, where

$$\Omega(t) = t^{-1}\psi_2(t) = t^{-1}[\varphi_2(t)+\delta\varphi_1(t)]$$
$$\beta'_1 = \frac{B_4}{B_3} - \frac{B_2}{B_1} + \frac{A_4}{A_3} - \frac{A_2}{A_1}\,; \quad \beta'_2 = \frac{B_2}{B_1} - \frac{B_4}{B_3} + \frac{A_4}{A_3} - \frac{A_2}{A_1} \tag{2.5}$$

and

$$C(t) = \frac{1}{\pi}\left\{\frac{1}{A_1}H_1(t) + \frac{1}{A_3}H_2(t) - \frac{1}{B_1}D_1(t) - \frac{1}{B_3}D_2(t)\right\}$$
$$\omega' = \frac{A_4}{A_3} - \frac{B_4}{B_3}, \quad A_4B_3 \neq A_3B_4\,. \tag{2.6}$$

It can be shown that $\beta'_1 = 0$ if δ satisfies the quadratic

$$A'\delta^2 + 2B'\delta + C' = 0\,, \tag{2.7}$$

where

$$A' = a_2a_4b_1b_4 - a_2a_3b_2b_4 - a_2a_4b_2b_3 + a_1a_4b_2b_4$$
$$B' = a_2a_3b_2b_3 - a_1a_4b_1b_4 \tag{2.8}$$
$$C' = a_1a_3b_1b_4 - a_1a_3b_2b_3 - a_2a_3b_1b_3 + a_1a_4b_1b_3\,.$$

The parameter δ being, in general, complex, causes the coefficients A_i, B_i, β'_2 and ω' to be complex.

3. The Solution of System I

The Wiener–Hopf equation mentioned in section 2 has been solved by Spence [8] for real coefficients. Spence's solution can, however, be modified to cover complex coefficients. The modified solution is presented here.

The integral equation under consideration is

$$\Omega(t) + \beta\int_0^\infty \Omega(\xi)k(t-\xi)d\xi - \lambda C(t) = \begin{cases} 0 & t>0 \\ c(t) & t<0 \end{cases} \tag{3.1}$$

where

$$\lambda = \frac{1}{\omega'}, \quad \beta = -\beta'_2\lambda\,. \tag{3.2}$$

Note that Spence's solution applies when $C(t)$ is of the form

$$C(t) = P\left(\frac{d}{dt}\right)k(t) + Q\left(\frac{d}{dt}\right)l(t)\,, \tag{3.3}$$

where P and Q are nth degree polynomials in the operator d/dt, and $l(t) = (1-\cos t)/\pi t$. The function $C(t)$ in (3.1) is of the form (3.3) if $f_1(\rho)$, $g_1(\rho)$ are polynomials and $f_2(\rho) = g_2(\rho) = 0$.

It can be shown that the solution of (3.1) is

$$\left.\begin{matrix}\Omega(t)\\ c(-t)\end{matrix}\right\} = \pm\frac{2}{\pi}\sinh\tfrac{1}{2}\pi\kappa\int_0^1 [U(w)\cos\varphi + wV(w)\sin\varphi]dw\,, \tag{3.4}$$

where $\varphi = \kappa\theta \pm wx$ (the $\pm$ sign corresponding respectively to $\Omega(t)$ and $c(-t)$), κ is complex and

$$U(w) = -\frac{\lambda}{2\beta}\left(\frac{2}{\pi}\sinh\tfrac{1}{2}\pi\kappa\right)\{B_1(w)\psi(w,\kappa)+iwB_2(w)\chi(w,\kappa)\} - \frac{\lambda}{2\beta}[s(w)+s(-w)] + \frac{\lambda}{2\beta}E(w), \tag{3.5}$$

$$V(w) = -\frac{\lambda i}{2\beta w}\left(\frac{2}{\pi}\sinh\tfrac{1}{2}\pi\kappa\right)\{B_2(w)\psi(w,\kappa)+iwB_1(w)\chi(w,\kappa)\} + \frac{\lambda i}{2\beta w}[s(w)-s(-w)] - \frac{\lambda i}{2\beta w}T(w), \tag{3.6}$$

where $\kappa=\sigma+i\tau=\pi^{-1}\log(1+\beta)$,

$$\left.\begin{matrix}\psi(w,\kappa)\\ \chi(w,\kappa)\end{matrix}\right\} = \operatorname{cosech}\tfrac{1}{2}\pi\kappa\int_0^{\frac{1}{2}\pi}\left\{\begin{matrix}\sinh\kappa y\cot y\\ \cosh\kappa y\end{matrix}\right. \frac{dy}{\cos^2 y+w^2\sin^2 y}, \tag{3.7}$$

$$\left.\begin{matrix}B_1(w)\\ B_2(w)\end{matrix}\right\} = Q(iw)\pm Q(-iw) \tag{3.8}$$

and

$$\left.\begin{matrix}E(w)\\ T(w)\end{matrix}\right\} = R(w)\pm R(-w). \tag{3.9}$$

The functions $R(w)$ in (3.9) and $s(w)$ in (3.5) and (3.6) are polynomials. The method for constructing them from $P(-iw)$ and $Q(-iw)$ is described by Spence. For second order polynomials

$$P\left(\frac{d}{dt}\right) = p_0+p_1\frac{d}{dt}+p_2\left(\frac{d}{dt}\right)^2; \quad Q\left(\frac{d}{dt}\right) = q_1\frac{d}{dt}+q_2\left(\frac{d}{dt}\right)^2, \tag{3.10}$$

$R(w)$ is found to be $R(w)=p_0+\kappa p_1+\frac{1}{2}\kappa^2 p_2-p_2 w^2-iw(p_1+\kappa p_2)$, or

$$R(w)=[A+p_2 w(\tau-w)+iB(\tau-w)], \tag{3.11}$$

where

$$A=p_0+\sigma p_1+\tfrac{1}{2}(\sigma^2-\tau^2)p_2; \quad B=p_1+\sigma p_2, \tag{3.12}$$

and $s(w)$ is found to be

$$s(w) = \frac{2}{\pi}\sinh\tfrac{1}{2}\kappa\pi(q_1 I_0+q_2 I_1-iwq_2 I_0), \tag{3.13}$$

where I_0 and I_1 are integrals defined in terms of $\psi(w,\kappa)$:

$$I_0=\kappa\psi(1,\kappa); \quad I_1=\tfrac{1}{2}-\tfrac{1}{2}\kappa I_0. \tag{3.14}$$

This completes the solution of the integral equation (3.1). The function $\Omega(t)=t^{-1}\{\varphi_2(t)+\delta\varphi_1(t)\}$ has thus been determined.

To obtain closed-form expressions for φ_1 and φ_2, one must write (3.4) twice, corresponding to δ_1 and δ_2, respectively (δ_1, δ_2 satisfy Eq. (2.7)). The resulting two equations lead at once to the solution of the original system of dual integral equations.

4. Existing Solutions

(a) *Reduction to Spence's Solution*

Spence [4] and Khadem [5] have studied the system described by (1.1) and (1.2), where $a_1=a_3=b_1=b_3=-b_4=1$, $a_2=\frac{1}{2}\beta$, $a_4=0$, $b_2=-(1+\frac{1}{2}\beta)$, $f_1(\rho)=a^{-1}w(\rho)$, $g_1(\rho)=a^{-1}u(\rho)$, $f_2(\rho)=g_2(\rho)=0$. In this case, δ has values of zero and -2.0. For $\delta=0$, we have

$$C(t) = \frac{2}{\pi}\int_0^1 [w^*(y)\cos yt - u^*(y)\sin yt]\,dy\,, \tag{4.1}$$

where

$$w^*(y) = \frac{1}{a}\frac{d}{dy}\int_0^y \frac{\rho w(\rho)d\rho}{(y^2-\rho^2)^{\frac{1}{2}}}\,, \quad u^*(y) = \frac{1}{a}\frac{1}{y}\frac{d}{dy}\int_0^y \frac{\rho^2 u(\rho)d\rho}{(y^2-\rho^2)^{\frac{1}{2}}}\,. \tag{4.2}$$

The integral equation (3.1) becomes

$$\Omega(t)+\beta\int_0^\infty \Omega(\xi)k(t-\xi)d\xi = C(t) \qquad t>0\,, \tag{4.3}$$

whose solution is given by (3.4). Note that $\kappa=\sigma=\pi^{-1}\log(1+\lambda)$, and $\tau=0$. The solution of (4.3) is

$$\left.\begin{matrix}\Omega(t)\\ -c(-t)\end{matrix}\right\} = \frac{2}{\pi\beta}\sinh\tfrac{1}{2}\kappa\pi\int_0^1 [U(w)\cos(\kappa\theta\pm wt)+wV(w)\sin(\kappa\theta\pm wt)]\,dw\,, \tag{4.4}$$

where

$$U(w) = A-p_2w^2 + \frac{2}{\pi}\sinh\tfrac{1}{2}\pi\kappa\,\{q_1[w^2\chi(w,\kappa)-\kappa\psi(1,\kappa)] + q_2[w^2\psi(w,\kappa)+\tfrac{1}{2}-\tfrac{1}{2}\kappa^2\psi(1,\kappa)]\} \tag{4.5}$$

$$V(w) = -B + \frac{2}{\pi}\sinh\tfrac{1}{2}\pi\kappa\,\{\psi(w,\kappa)q_1 - q_2[w^2\chi(w,\kappa)-\kappa\psi(1,\kappa)]\}\,.$$

$$\begin{aligned} B_1(w) &= -2q_2w^2 & B_2(w) &= 2q_1 iw \\ E(w) &= 2A-p_2w^2 & T(w) &= -2iwB \end{aligned} \tag{4.6}$$

$$\begin{aligned} s(w)+s(-w) &= \frac{4}{\pi}\sinh\tfrac{1}{2}\pi\kappa\,\{\kappa\psi(1,\kappa)q_1-[\tfrac{1}{2}-\tfrac{1}{2}\kappa^2\psi(1,\kappa)]q_2\} \\ s(w)-s(-w) &= -\frac{4}{\pi}iw\sinh\tfrac{1}{2}\pi\kappa\,\{\kappa q_2\psi(1,\kappa)\}\,, \end{aligned} \tag{4.7}$$

Equations (4.4) and (4.5) are in agreement with Spence [4].

(b) *Reductions to Gladwell's Solution*

Gladwell [6] has recently considered three pairs of simultaneous dual integral equations which he reduced to two pairs of the form considered here, with $a_1=b_2=1$, $a_2=b_1=0$, $a_3=b_4=\alpha$, $a_4=b_3=1-\alpha$, $f_1(\rho)=d_1$, $g_1(\rho)=\nu\rho$, $f_2(\rho)=g_2(\rho)=0$. These values of a_i, b_i give $\delta=\pm1$. For $\delta=+1$,

$$\begin{aligned} &A_1=1; \quad A_2=0; \quad A_3=2\alpha-1; \quad A_4=1-\alpha \\ &B_1=-1; \quad B_2=1; \quad B_3=1-2\alpha; \quad B_4=\alpha \end{aligned} \tag{4.8}$$

$$\beta_2' = 2(1-\alpha)/(2\alpha-1)\,, \quad \omega' = 1/(2\alpha-1)\,. \tag{4.9}$$

At this stage it is convenient to define quantities σ and η such that

$$e^{2\eta} = 1/(2\alpha-1)\,, \quad \sigma = 2\eta/\pi\,. \tag{4.10}$$

Then

$$\omega' = e^{2\eta}\,, \quad \beta = -2(1+\alpha)\;; \quad 1+\beta = e^{-2\eta}\,. \tag{4.11}$$

Since $1+\beta$ is real, $\kappa=\pi^{-1}\log(1+\beta)=-\sigma$, also

$$C(t) = 2d_1 k(t) - 4v \frac{dk}{dt}, \tag{4.12}$$

that is,

$$\begin{aligned} p_0 = 2d_1, \quad p_1 = -4v, \\ p_2 = p_3 = \dots p_n = 0, \quad q_1 = q_2 = \dots q_n = 0. \end{aligned} \tag{4.13}$$

The solution is

$$\varphi_2(t) + \varphi_1(t) = \frac{2}{\pi} \sinh \tfrac{1}{2}\sigma\pi \int_0^1 [U(w) \cos(\sigma\theta - wt) - wV(w) \sin(\sigma\theta - wt)] dw, \tag{4.14}$$

where

$$U(w) = 4(d_1 + 2v\sigma) e^{-2\eta}, \quad V(w) = 8ve^{-2\eta},$$

and since for $\kappa = -\sigma$, $\beta^{-1} \sinh \frac{1}{2}\sigma\pi = -\frac{1}{2}e^{\eta}$, then

$$\varphi_2(t) + \varphi_1(t) = \frac{2}{\pi} e^{-\eta} \int_0^1 [(d_1 + 2\sigma v) \cos(\sigma\theta - wt) - 2vw \sin(\sigma\theta - wt)], \tag{4.15}$$

corresponding to $\delta = -1$, Eq. (3.1) can be solved again (note that in this case (4.8)–(4.14) should be altered accordingly), to give

$$\varphi_2(t) - \varphi_1(t) = -\frac{2}{\pi} e^{\eta} \int_0^1 [(d_1 + 2\sigma v) \cos(\sigma\theta + wt) - 2vw \sin(\sigma\theta + wt)] dw, \tag{4.16}$$

in agreement with the result quoted by Gladwell.

REFERENCES

[1] F. Erdogan and L. Y. Bahar, On the solution of simultaneous dual integral equations, *J. Soc. Indust. Appl. Math.*, 12 (1964).

[2] R. A. Westmann, Simultaneous pairs of dual integral equations, *SIAM Review*, 7, 3 (1965).

[3] L. M. Keer, Coupled pairs of dual integral equations, *Quart. Appl. Math.*, 25 (1968) 453–457.

[4] D. A. Spence, Self similar solutions to adhesive contact problems with incremental loading, *Proc. Roy Soc.*, A305 (1968) 5–80.

[5] R. Khadem, D. Phil. Thesis, Oxford University, Part III, Sept. 1967.

[6] G. M. L. Gladwell, A contact problem for a circular cylindrical punch in adhesive contact with an elastic half-space; the case of rocking, and translation parallel to the plane, *Int. J. Engng. Sci.*, 7 (1969) 295–307.

[7] B. Noble, The solution of Bessel function dual integral equations by a multiplying factor method, *Proc. Camb. Phil. Soc.*, 59 (1963).

[8] D. A. Spence, A Wiener–Hopf equation arising in elastic contact problems, *Proc. Roy. Soc.*, A305 (1968) 81–92.